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A lattice is said to be semi-complete provided every non-empty subset has a least upper bound. A function from one lattice into another is said to be a similarity mapping provided it is one-to-one, onto, and preserves least upper bounds. <u>Theorem</u>. Two semi-complete lattices are lattice isomorphic if and only if they are similar. The author has shown that among the topological properties preserved by lattice isomorphisms are compact, connected, locally compact, locally connected, and second countable. In addition, the author has given examples to show that separable and first countable are not preserved by lattice isomorphisms.

LATTICE ISOMORPHISMS

by

Karen Carter Lamb

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INTRODUCTION

The purpose of this thesis is to investigate the preservation of topological properties under lattice isomorphisms. The idea of using the usual lattice structure on topologies was originated by Birkhoff in [1], and has been studied by Thron in [2], [6], and [7] and by Kerstan in [5]. The idea of similarity was first introduced by Goolsby in [4], and many of the ideas of this thesis originated in that paper. The ideas in Chapter V were originated by the author, and answered the two questions in [4]. The author assumes a working knowledge of set theory, and, in particular, the well-ordering property. The reader is referred to [1], [3] and [7] for definitions and results not covered in this thesis.

In Chapter I, the definitions of lattice, lattice isomorphism, and similarity are introduced and basic theorems concerning these concepts are proved.

In Chapter II, the ideas of Chapter I are specialized to topologies and the basic results used in the remainder of the paper are stated and proved.

In Chapters III and IV it is shown that compact, locally compact, connected and locally connected are preserved by lattice isomorphisms.

In Chapter V, in addition to showing that second countable is preserved by lattice isomorphisms, an example is given to show that separable and first countable need not be preserved.

In Chapter VI, it is shown that T_1-spaces are homeomorphic if and only if their topologies are lattice isomorphic.

CHAPTER I

Definition 1: To say that the ordered pair (X,R) is a partially ordered set means that X is a set, and R is a subset of $X \times X$ such that:

1. If $a \in X$, then $(a,a) \in R$,
2. If $a, b \in X$, $(a,b) \in R$, and $(b,a) \in R$, then $a = b$, and
3. If $a, b, c \in X$, $(a,b) \in R$, and $(b,c) \in R$, then $(a,c) \in R$.

If $a, b \in X$, then to say that a is related to b by R, denoted by aRb, means that $(a,b) \in R$.

Definition 2: Let $(X,<)$ be a partially ordered set, let $A \subset X$, and let $b \in X$. To say that b is an upper bound of A means that if $a \in A$, then $a < b$. To say that b is a least upper bound of A, denoted by $b = \text{l.u.b.}A$, means that b is an upper bound of A and if h is an upper bound of A, $b < h$.

Definition 3: Let $(X,<)$ be a partially ordered set, let $A \subset X$, and let $b \in X$. To say that b is a lower bound of A means that if $a \in A$, then $b < a$. To say that b is a greatest lower bound of A, denoted by $b = \text{g.l.b.}A$, means that b is a lower bound of A and if h is a lower bound of A, $h < b$.

Theorem 1: Let $(X,<)$ be a partially ordered set, and let $A \subset X$. Then A has at most one least upper bound, and at most one greatest lower bound.

<u>Proof</u>: Suppose A has two least upper bounds, b and c. Then $b < c$ and $c < b$. So $b = c$. Therefore, A has at most one least upper bound. Similarly, A has at most one greatest lower bound.

<u>Definition 4</u>: Let $(X,<)$ be a partially ordered set. To say that $(X,<)$ is a <u>lattice</u> means that if $A \subset X$, and $A \neq \phi$, and A is finite, then A has a least upper bound and a greatest lower bound.

<u>Definition 5</u>: Let $(X,<)$ be a lattice. To say that $(X,<)$ is a <u>semi-complete</u> <u>lattice</u> means that if $A \subset X$, and $A \neq \phi$, then A has a least upper bound.

<u>Definition 6</u>: Let $(K,<)$ and $(L,<<)$ be lattices. To say that a function f is a <u>lattice</u> <u>isomorphism</u> <u>from</u> $(K,<)$ <u>onto</u> $(L,<<)$ means:

1. The function f is one-to-one,
2. The function f has domain K and range L,
3. If $a, b \in K$ and $a < b$, then $f(a) << f(b)$, and
4. If $c, d \in L$ and $c << d$, then $f^{-1}(c) < f^{-1}(d)$.

To say that $(K,<)$ is <u>lattice</u> <u>isomorphic</u> <u>to</u> $(L,<<)$ means that there exists a lattice isomorphism from $(K,<)$ onto $(L,<<)$.

<u>Definition 7</u>: Let $(K,<)$ and $(L,<<)$ be semi-complete lattices. To say that a function f is a <u>similarity</u> <u>mapping</u> <u>from</u> $(K,<)$ <u>onto</u> $(L,<<)$ means:

1. The function f is one-to-one,
2. The function f has domain K and range L, and
3. If $A \subset K$ and $A \neq \phi$, then

   $F(\text{l.u.b.}A) = \text{l.u.b.}\{F(a) \mid a \in A\}$.

To say that $(K,<)$ is <u>similar to</u> $(L,<<)$ means that there exists a similarity mapping from $(K,<)$ onto $(L,<<)$.

<u>Theorem 2</u>: Let $(K,<)$ and $(L,<<)$ be semi-complete lattices and let F be a similarity mapping from $(K,<)$ onto $(L,<<)$. Then F^{-1} is a similarity mapping from $(L,<<)$ onto $(K,<)$.

<u>Proof</u>: Since F is one-to-one and since F has domain K and range L, then clearly F^{-1} is one-to-one and has domain L and range K. Now let $B \subset L$ such that $B \neq \phi$. Let $A = \{F^{-1}(b) \mid b \in B\}$. Then $A \subset K$ and $A \neq \phi$. Then

$$F^{-1}(\text{l.u.b.}B) = F^{-1}(\text{l.u.b.}\{b \mid b \in B\}) =$$
$$F^{-1}(\text{l.u.b.}\{F(F^{-1}(b)) \mid b \in B\}) = F^{-1}(\text{l.u.b.}\{F(a) \mid a \in A\}) =$$
$$F^{-1}(F(\text{l.u.b.}\{a \mid a \in A\})) = \text{l.u.b.}\{a \mid a \in A\} =$$
$$\text{l.u.b.}\{F^{-1}(b) \mid b \in B\}.$$

Thus, F^{-1} is a similarity mapping from $(L,<<)$ onto $(K,<)$.

<u>Theorem 3</u>: Let $(K,<)$ and $(L,<<)$ be semi-complete lattices and let F be a similarity mapping from $(K,<)$ onto $(L,<<)$. Then if $a, b \in K$ with $a < b$, then $F(a) << F(b)$.

<u>Proof</u>: Let $a, b \in K$ with $a < b$. Then $b = \text{l.u.b.}\{a,b\}$. Hence, $F(b) = F(\text{l.u.b.}\{a,b\}) = \text{l.u.b.}\{F(a), F(b)\}$. Therefore, $F(a) << F(b)$.

<u>Theorem 4</u>: Let $(K,<)$ and $(L,<<)$ be semi-complete lattices and let F be a similarity mapping from $(K,<)$ onto $(L,<<)$. Then if $A \subset K$ and $A \neq \phi$ and A is finite, then

$F(\text{g.l.b.}A) = \text{g.l.b.}\{F(a) \mid a \in A\}$.

Proof: Let A be a non-empty finite subset of K. Then the g.l.b.A exists and for each $a \in A$, g.l.b.$A < a$. So by Theorem 3, for each $a \in A$, $F(\text{g.l.b.}A) < F(a)$. So $F(\text{g.l.b.}A)$ is a lower bound of $\{F(a) \mid a \in A\}$. So $F(\text{g.l.b.}A) << \text{g.l.b.}\{F(a) \mid a \in A\}$.

Now by Theorem 2, F^{-1} is a similarity mapping from $(L,<<)$ onto $(K,<)$. Since A is a non-empty finite subset of K, then $\{F(a) \mid a \in A\}$ is a non-empty finite subset of L. Thus $F^{-1}(\text{g.l.b.}\{F(a) \mid a \in A\}) < \text{g.l.b.}\{F^{-1}(F(a)) \mid a \in A\}$. By Theorem 3, $F(F^{-1}(\text{g.l.b.}\{F(a) \mid a \in A\})) << F(\text{g.l.b.}\{F^{-1}(F(a)) \mid a \in A\})$. Thus

$$\text{g.l.b.}\{F(a) \mid a \in A\} = F(F^{-1}(\text{g.l.b.}\{F(a) \mid a \in A\})) <<$$
$$F(\text{g.l.b.}\{F^{-1}(F(a)) \mid a \in A\}) = F(\text{g.l.b.}\{a \mid a \in A\}) = F(\text{g.l.b.}A).$$

Therefore, $F(\text{g.l.b.}A) = \text{g.l.b.}\{F(a) \mid a \in A\}$.

Theorem 5: Let $(K,<)$ and $(L,<<)$ be semi-complete lattices, and let F be a one-to-one function with domain K and range L. Then F is a lattice isomorphism from $(K,<)$ onto $(L,<<)$ if and only if F is a similarity mapping from $(K,<)$ onto $(L,<<)$.

Proof: Suppose F is a similarity mapping from $(K,<)$ onto $(L,<<)$. Let $a, b \in K$ with $a < b$. Then by Theorem 3, $F(a) << F(b)$. Let $c, d \in L$ with $c << d$. Since F^{-1} is a similarity mapping from $(L,<<)$ onto $(K,<)$, then by Theorem 3, $F^{-1}(c) < F^{-1}(d)$. Hence, F is a lattice isomorphism from $(K,<)$ onto $(L,<<)$.

Suppose F is a lattice isomorphism from $(K,<)$ onto $(L,<<)$. Let $A \subset K$ such that $A \neq \phi$. Then l.u.b.A exists and for each $a \in A$, $a <$ l.u.b.A. So for each $a \in A$, $F(a) << F(\text{l.u.b.}A)$. Thus,

$F(\text{l.u.b.}A)$ is an upper bound of $\{F(a) \mid a \in A\}$. So $\text{l.u.b.}\{F(a) \mid a \in A\} << F(\text{l.u.b.}A)$. Since $A \neq \phi$, $\{F(a) \mid a \in A\} \neq \phi$. Then $\text{l.u.b.}\{F(a) \mid a \in A\}$ exists and for each $a \in A$, $F(a) << \text{l.u.b.}\{F(a) \mid a \in A\}$. Hence, for each $a \in A$, $a = F^{-1}(F(a)) < F^{-1}(\text{l.u.b.}\{F(a) \mid a \in A\})$. Therefore, $F^{-1}(\text{l.u.b.}\{F(a) \mid a \in A\})$ is an upper bound of A. So $\text{l.u.b.}A < F^{-1}(\text{l.u.b.}\{F(a) \mid a \in A\})$. Thus, $F(\text{l.u.b.}A) << F(F^{-1}(\text{l.u.b.}\{F(a) \mid a \in A\})) = \text{l.u.b.}\{F(a) \mid a \in A\}$.

So $F(\text{l.u.b.}A) = \text{l.u.b.}\{F(a) \mid a \in A\}$. Hence, F is a similarity mapping from $(K,<)$ onto $(L,<<)$.

CHAPTER II

Definition 8: Let X be a set and let $\mathcal{U}$ be a collection of subsets of X. Then $\cup\,\mathcal{U}$ is defined to be $\{x \mid \text{there exists a } U \in \mathcal{U} \text{ such that } x \in \mathcal{U}\}$ and $\cap\,\mathcal{U}$ is defined to be $\{x \mid \text{if } U \in \mathcal{U}, \text{ then } x \in U\}$.

Definition 9: Let X be a set and let T be a collection of subsets of X. To say that T is a topology for X means that:

1. The empty set is an element of T,
2. The set X is an element of T,
3. If $\mathcal{U} \subset T$, then $\cup\,\mathcal{U} \in T$, and
4. If $\mathcal{U} \subset T$ and $\mathcal{U}$ is finite, then $\cap\,\mathcal{U} \in T$.

To say that (X,T) is a topological space means that X is a set and T is a topology for X.

Definition 10: Let (X,T) be a topological space. To say that U is open means that $U \in T$. To say that C is closed means that $X - C$ is open. The collection of all closed subsets of X is dentoed by C_T.

Definition 11: Let (X,T) be a topological space, let $A \subset X$, and let $p \in X$. To say that p is a limit point of A means that if $U \in T$ and $p \in U$, there is an element a of A such that $p \neq a$ and $a \in U$.

Definition 12: Let (X,T) be a topological space and let $A \subset X$. Then the closure of A, denoted $\bar{A}$, is defined to be the union of A with the set of all limit points of A.

Definition 13: Let (X,T) be a topological space. The usual lattice structure $<$ on T is defined by if $U, V \in T$, then $U < V$ if and only if $U \subset V$. The usual lattice structure $<^*$ on C_T is defined by if $C, D \in C_T$, then $C <^* D$ if and only if $D \subset C$.

Theorem 6: Let (X,T) be a topological space and let $<$ and $<^*$ be the usual lattice structures on T and C_T respectively. Then $(T,<)$ and $(C_T,<^*)$ are partially ordered sets.

Proof: Clearly, T is a set and $<$ is a subset of $X \times X$. Now if $U \in T$, then $U \subset U$ so $U < U$. If $U, V \in T$ and $U < V$ and $V < U$, then $U \subset V$ and $V \subset U$; hence $U = V$. If $U, V, W \in T$, $U < V$, and $V < W$, then $U \subset V$ and $V \subset W$ so $U \subset W$; hence $U < W$. Therefore, $(T,<)$ is a partially ordered set.

Similarly, $(C_T,<^*)$ is a partially ordered set.

Theorem 7: Let (X,T) be a topological space, and let $<$, $<^*$ be the usual lattice structures on T and C_T respectively. Then if $\mathcal{U}$ is a non-empty subset of T, l.u.b.$\mathcal{U} = \cup\,\mathcal{U}$, and g.l.b.$\mathcal{U} = \cap\,\mathcal{U}$. Then, also, if $\mathcal{C}$ is a non-empty subset of C_T, l.u.b.$\mathcal{C} = \cap\,\mathcal{C}$, and g.l.b.$\mathcal{C} = \cup\,\mathcal{C}$.

Proof: Suppose $\mathcal{U}$ is a non-empty subset of T. Then if $U \in \mathcal{U}$, $U \subset \cup\,\mathcal{U}$ so $U < \cup\,\mathcal{U}$. Hence, $\cup\,\mathcal{U}$ is an upper bound of $\mathcal{U}$.

Now let V be an upper bound of $\mathcal{U}$. Then $V \in T$ and for all $U \in \mathcal{U}$, then $U < V$, so $U \subset V$. Therefore $\cup\, \mathcal{U} \subset V$, so $\cup\, \mathcal{U} < V$. Hence $\cup\, \mathcal{U} = \text{l.u.b.}\mathcal{U}$.

If $U \in \mathcal{U}$, then $\cap\, \mathcal{U} \subset U$, so $\cap\, \mathcal{U} < U$. Hence, $\cap\, \mathcal{U}$ is a lower bound of $\mathcal{U}$. Now let W be a lower bound of $\mathcal{U}$. Then $W \in T$ and for all $U \in \mathcal{U}$, $W < U$ so $W \subset U$. Therefore $W \subset \cap\, \mathcal{U}$, so $W < \cap\, \mathcal{U}$. Hence $\cap\, \mathcal{U} = \text{g.l.b.}\mathcal{U}$.

Similarly, if $\mathcal{C}$ is a non-empty subset of C_T, then $\text{l.u.b.}\mathcal{C} = \cap\, \mathcal{C}$, and $\text{g.l.b.}\mathcal{C} = \cup\, \mathcal{C}$.

Theorem 8: Let (X,T) be a topological space, and let $<$, $<^*$ be the usual lattice structures on T and C_T respectively. Then $(T,<)$ and $(C_T,<^*)$ are semi-complete lattices.

Proof: Suppose $\mathcal{U} \subset T$ and $\mathcal{U} \neq \phi$. Then by Theorem 7, $\text{l.u.b.}\mathcal{U} = \cup\, \mathcal{U} \in T$. So if $\mathcal{U} \subset T$ and $\mathcal{U} \neq \phi$, then $\mathcal{U}$ has a least upper bound.

Suppose $\mathcal{U}$ is finite. Then by Theorem 7, $\text{g.l.b.}\mathcal{U} = \cap\, \mathcal{U} \in T$. So if $\mathcal{U} \subset T$ and $\mathcal{U} \neq \phi$ and $\mathcal{U}$ is finite, then $\mathcal{U}$ has a greatest lower bound.

Thus, $(T,<)$ is a semi-complete lattice.

Similarly, $(C_T,<^*)$ is a semi-complete lattice.

Theorem 9: Let (X,S) and (Y,T) be topological spaces, let $<$ and $<<$ be the usual lattice structures on S and T respectively, and let F be a lattice isomorphism from $(S,<)$ onto $(T,<<)$. Then $F(\phi) = \phi$ and $F(X) = Y$.

Proof: Since F is onto, there is an $A \in S$ such that $F(A) = \phi$. Now $\phi \subset A$ and $\phi, A \in S$, so $\phi < A$. Hence $F(\phi) << F(A)$. Since $F(A) = \phi$, then $F(\phi) << \phi$. Now since $\phi \in S$, there exists a $B \in T$ such that $F(\phi) = B$. But $\phi \subset B$, so $\phi << B$; hence, $\phi << F(\phi)$. Therefore, $\phi = F(\phi)$.

Since F is onto, there is a $C \in S$ so that $F(C) = Y$. Now $C \subset X$, so $C < X$; hence $F(C) << F(X)$. Since $Y = F(C)$, then $Y << F(X)$. Since $X \in S$, there exists a $D \in T$ such that $F(X) = D$. But $D \subset Y$, so $D << Y$; hence $F(X) << Y$. Therefore, $F(X) = Y$.

Theorem 10: Let (X,S) and (Y,T) be topological spaces, let $<$ and $<<$ be the usual lattice structures on S and T respectively, and let F be a lattice isomorphism from $(S,<)$ onto $(T,<<)$. Then

1. If $\mathcal{U} \subset S$, $F(\cup\{U \mid U \in \mathcal{U}\}) = \cup\{F(U) \mid U \in \mathcal{U}\}$, and
2. If $\mathcal{U} \subset S$ and $\mathcal{U}$ is finite,
   $F(\cap\{U \mid U \in \mathcal{U}\}) = \cap\{F(U) \mid U \in \mathcal{U}\}$.

Proof: If $\mathcal{U} = \phi$, then

$F(\cup\{U \mid U \in \mathcal{U}\}) = F(\cup\,\phi) = F(\phi) = \phi = \cup\,\phi = \cup\{F(U) \mid U \in \mathcal{U}\}$.

Suppose $\mathcal{U} \subset S$ with $\mathcal{U} \neq \phi$. By Theorem 5, F is a similarity mapping from $(S,<)$ onto $(T,<<)$. And by Theorem 7, $\text{l.u.b.}\,\mathcal{U} = \cup\,\mathcal{U}$ and $\text{l.u.b.}\{F(U) \mid U \in \mathcal{U}\} = \cup\{F(U) \mid U \in \mathcal{U}\}$. Hence,

$$F(\cup\{U \mid U \in \mathcal{U}\}) = F(\cup\,\mathcal{U}) = F(\text{l.u.b.}\,\mathcal{U}) =$$
$$\text{l.u.b.}\{F(U) \mid U \in \mathcal{U}\} = \cup\{F(U) \mid U \in \mathcal{U}\}.$$

If $\mathcal{U} = \phi$, then

$F(\cap\{U \mid U \in \mathcal{U}\}) = F(\cap\,\phi) = F(X) = Y = \cap\,\phi = \cap\{F(U) \mid U \in \mathcal{U}\}$.

Suppose $\mathcal{U}$ is finite. By Theorem 7, g.l.b.$\mathcal{U} = \cap\, \mathcal{U}$ and g.l.b.$\{F(U) \mid U \in \mathcal{U}\} = \cap\, \{F(U) \mid U \in \mathcal{U}\}$. By Theorem 4, $F(\text{g.l.b.}\mathcal{U}) = \text{g.l.b.}\{F(U) \mid U \in \mathcal{U}\}$. Then

$$F(\cap\, \{U \mid U \in \mathcal{U}\}) = F(\cap\, \mathcal{U}) = F(\text{g.l.b.}\mathcal{U}) =$$
$$\text{g.l.b.}\{F(U) \mid U \in \mathcal{U}\} = \cap\, \{F(U) \mid U \in \mathcal{U}\}.$$

Definition 14: Let (X,S) and (Y,T) be topological spaces, let $<$ and $<<$ be the usual lattice structures on S and T respectively, let $<*$ and $<<*$ be the usual lattice structures on C_S and C_T respectively, and let F be a lattice isomorphism from $(S,<)$ onto $(T,<<)$. The dual lattice isomorphism of F, denoted $\mathcal{D}_F$, is defined by if $C \in C_S$, then $\mathcal{D}_F(C) = Y - F(X - C)$.

Theorem 11: Let (X,S) and (Y,T) be topological spaces, let $<$, $<<$, $<*$, $<<*$, be the usual lattice structures on S, T, C_S and C_T respectively, let F be a lattice isomorphism from $(S,<)$ onto $(T,<<)$. Then $\mathcal{D}_F$ is one-to-one, has domain C_S and range C_T, and $\mathcal{D}_F{}^{-1} = \mathcal{D}_{F^{-1}}$.

Proof: Let $C, K \in C_S$ with $\mathcal{D}_F(C) = \mathcal{D}_F(K)$. Then

$$C = X - (X - C) = X - [F^{-1}(F(X - C))] =$$
$$X - [F^{-1}(Y - (Y - F(X - C)))] = X - [F^{-1}(Y - \mathcal{D}_F(C))] =$$
$$X - [F^{-1}(Y - \mathcal{D}_F(K))] = X - [F^{-1}(Y - (Y - F(X - K)))] =$$
$$X - [F^{-1}(F(X - K))] = X - (X - K) = K.$$

So $\mathcal{D}_F$ is one-to-one.

Clearly, $\mathcal{D}_F$ has domain C_S. Let $D \in C_T$. Then $(Y - D) \in T$ and thus $F^{-1}(Y - D) \in S$; hence $X - F^{-1}(Y - D) \in C_S$. And,

$$\mathcal{D}_F[X - F^{-1}(Y - D)] = Y - F[X - (X - F^{-1}(Y - D))] =$$
$$Y - [F(F^{-1}(Y - D))] = Y - (Y - D) = D.$$

Hence, $\mathcal{D}_F$ has range C_T.

Let $E \in C_T$. Then there exists exactly one $A \in C_S$ such that $\mathcal{D}_F(A) = E$. Thus, $\mathcal{D}_F^{-1}(E) = A$. Now $E = \mathcal{D}_F(A) = Y - F(X - A)$. So, $Y - E = F(X - A)$, and $F^{-1}(Y - E) = F^{-1}(F(X - A)) = X - A$, so $X - F^{-1}(Y - E) = A$. Hence, $\mathcal{D}_F^{-1}(E) = A = X - F^{-1}(Y - E) = \mathcal{D}_{F^{-1}}(E)$. Therefore, $\mathcal{D}_F^{-1} = \mathcal{D}_{F^{-1}}$.

<u>Theorem 12</u>: Let (X,S) and (Y,T) be topological spaces, let $<$, $<<$, $<*$, $<<*$ be the usual lattice structures on S, T, C_S and C_T respectively, let F be a lattice isomorphism from $(S,<)$ onto $(T,<<)$. Then $\mathcal{D}_F$ is a lattice isomorphism from $(C_S,<*)$ onto $(C_T,<<*)$.

<u>Proof</u>: By Theorem 11, $\mathcal{D}_F$ is one-to-one, and has domain C_S and range C_T.

Let $C, K \in C_S$ with $C <* K$. Then $K \subset C$, so $X - C \subset X - K$, or $X - C < X - K$. Thus $F(X - C) << F(X - K)$, or $F(X - C) \subset F(X - K)$, so $Y - F(X - K) \subset Y - F(X - C)$. Hence, $\mathcal{D}_F(C) = Y - F(X - C) <<* Y - F(X - K) = \mathcal{D}_F(K)$.

Let $D, E \in C_T$ with $D <<* E$. Then $E \subset D$, so $Y - D \subset Y - E$, or $Y - D << Y - E$. Thus $F^{-1}(Y - D) < F^{-1}(Y - E)$, or $F^{-1}(Y - D) \subset F^{-1}(Y - E)$, so $X - F^{-1}(Y - E) \subset X - F^{-1}(Y - D)$. Hence, $\mathcal{D}_F^{-1}(D) = \mathcal{D}_{F^{-1}}(D) = X - F^{-1}(Y - D) <* X - F^{-1}(Y - E) = \mathcal{D}_{F^{-1}}(E) = \mathcal{D}_F^{-1}(E)$.

Therefore, $\mathcal{D}_F$ is a lattice isomorphism from $(C_S,<*)$ onto $(C_T,<<*)$.

Throughout the remainder of this paper, the usual lattice structure will be assumed, and unions and intersections will be used instead of least upper bounds and greatest lower bounds. In order to facilitate the reading of this paper, the frequently used facts which follow from the previous theorems in Chapter II, are now listed.

If (X,S) and (Y,T) are topological spaces and F is a lattice isomorphism from S onto T, then:

1. The function F is one-to-one, has domain S and range T,
2. The function $\mathcal{D}_F$ is one-to-one, has domain C_S and range C_T,
3. If $U_1, U_2 \in S$ and $U_1 \subset U_2$, then $F(U_1) \subset F(U_2)$,
4. If $C_1, C_2 \in C_S$ and $C_1 \subset C_2$, then $\mathcal{D}_F(C_1) \subset \mathcal{D}_F(C_2)$,
5. If $V_1, V_2 \in T$ and $V_1 \subset V_2$, then $F^{-1}(V_1) \subset F^{-1}(V_2)$,
6. If $D_1, D_2 \in C_T$ and $D_1 \subset D_2$, then $\mathcal{D}_F^{-1}(D_1) \subset \mathcal{D}_F^{-1}(D_2)$,
7. If $\mathcal{U} \subset S$, then $F(\cup \{U \mid U \in \mathcal{U}\}) = \cup \{F(U) \mid U \in \mathcal{U}\}$,
8. If $\mathcal{C} \subset C_S$, then $\mathcal{D}_F(\cap \{C \mid C \in \mathcal{C}\}) = \cap \{\mathcal{D}_F(C) \mid C \in \mathcal{C}\}$,
9. If $\mathcal{V} \subset T$, then $F^{-1}(\cup \{V \mid V \in \mathcal{V}\}) = \cup \{F^{-1}(V) \mid V \in \mathcal{V}\}$,
10. If $\mathcal{D} \subset C_T$, then $\mathcal{D}_F^{-1}(\cap \{D \mid D \in \mathcal{D}\}) = \cap \{\mathcal{D}_F^{-1}(D) \mid D \in \mathcal{D}\}$,
11. If $\mathcal{U} \subset S$ and $\mathcal{U}$ is finite, then
$F(\cap \{U \mid U \in \mathcal{U}\}) = \cap \{F(U) \mid U \in \mathcal{U}\}$,
12. If $\mathcal{C} \subset C_S$ and $\mathcal{C}$ is finite, then
$\mathcal{D}_F(\cup \{C \mid C \in \mathcal{C}\}) = \cup \{\mathcal{D}_F(C) \mid C \in \mathcal{C}\}$,

13. If $\mathcal{V} \subset T$ and $\mathcal{V}$ is finite, then

$F^{-1}(\cap \{V \mid V \in \mathcal{V}\}) = \cap \{F^{-1}(V) \mid V \in \mathcal{V}\}$,

14. If $\mathcal{D} \subset C_T$ and $\mathcal{D}$ is finite, then

$\mathcal{D}_F^{-1}(\cup \{D \mid D \in \mathcal{D}\}) = \cup \{\mathcal{D}_F^{-1}(D) \mid D \in \mathcal{D}\}$,

15. $F(\phi) = \phi$, $F^{-1}(\phi) = \phi$, $F(X) = Y$, and $F^{-1}(Y) = X$, and

16. $\mathcal{D}_F(\phi) = \phi$, $\mathcal{D}_F^{-1}(\phi) = \phi$, $\mathcal{D}_F(X) = Y$, and $\mathcal{D}_F^{-1}(Y) = X$.

Theorem 13: Let (X,S) and (Y,T) be topological spaces, and let F be a lattice isomorphism from S onto T. Let $U \in S$, let $V = F(U)$ and let $p \in V$. Then there is an $x \in U$ such that if $W \in S$ and $x \in W$, then $p \in F(W)$.

Proof: Suppose that for every $x \in U$, there is an $\mathcal{O} \in S$ such that $x \in \mathcal{O}$ and $p \notin F(\mathcal{O})$. For each $x \in U$, let U_x be an element of S such that $x \in U_x$ and $p \notin F(U_x)$, and let $V_x = U_x \cap U$. Now for each $x \in U$, $p \notin F(U_x)$ and $F(V_x) \subset F(U_x)$. Thus, $F(U) = F(\cup \{V_x \mid x \in U\}) = \cup \{F(V_x) \mid x \in U\} \nsupseteq V$. But $F(U) = V$, which is impossible. Hence, there is an $x \in U$ such that if $W \in S$ and $x \in W$, then $p \in F(W)$.

Theorem 14: Let (X,T) be a topological space, and let $C \subset X$ such that $\{p \mid p$ is a limit point of $C\} \subset C$. Then $C \in C_T$.

Proof: Let $x \in X - C$. Then x is not a limit point of C. Thus, there exists a $U_x \in T$ such that $x \in U_x$ and $U_x \cap C = \phi$.

Let $U = \cup\{U_a \mid a \in X - C\}$. Then $U \in T$.

Let $y \in U$. Then there exists a $U_y \in U$ such that $y \in U_y$; $U_y \cap C = \phi$, therefore $U_y \subset X - C$, so $y \in X - C$, which means $U \subset X - C$.

Let $z \in X - C$. Then $z \in U_z \subset U$. Thus, $z \in U$, which means $X - C \subset U$.

Therefore, $X - C = U \in T$. Thus $C = X - (X - C) \in C_T$.

Theorem 15: Let (X,T) be a topological space, and let $M \subset X$. Then $\bar{M} \in C_T$.

Proof: Let p be a limit point of $\bar{M}$. Assume $p \notin \bar{M}$. Let $U \in T$ such that $p \in U$. Since p is a limit point of $\bar{M}$, there exists a $q \in \bar{M}$ such that $p \neq q$ and $q \in U$. Either $q \in M$ or q is a limit point of M. If $q \in M$, then there is an element of M, namely q, such that $p \neq q$ and $q \in U$. If q is a limit point of M, since $U \in T$ and $q \in U$, then there exists an $m \in M$ such that $q \neq m$ and $m \in U$; since $m \in M$ and $p \notin M$, then $m \neq p$, and once again there exists an element of M, namely m, such that $p \neq m$ and $m \in U$. Thus, p is a limit point of M, which means $p \in \bar{M}$, which is a contradiction. Thus, $p \in \bar{M}$. So $\{p \mid p$ is a limit point of $\bar{M}\} \subset \bar{M}$. Hence, by Theorem 14, $\bar{M} \in C_T$.

Theorem 16: Let (X,T) be a topological space, let $M \in C_T$, and let $A \subset M$. Then $\bar{A} \subset M$.

<u>Proof</u>: Let p be a limit point of A. Suppose $p \in (X - M) \in T$. Then there is an $a \in A$ such that $p \neq a$ and $a \in (X - M)$. But $a \in A \subset M$ so $a \in M$. This is impossible; thus, $p \in M$. Therefore if x is a limit point of A, then $x \in M$, and if $x \in A$, then $x \in M$. Hence $\bar{A} \subset M$.

<u>Corollary 16.1</u>: Let (X,T) be a topological space, and let $U, V \in T$ such that $U \cap V = \phi$. Then $\bar{U} \cap V = \phi$.

<u>Proof</u>: Since $U \cap V = \phi$, then $U \subset (X - V)$. Since $V \in T$, $(X - V) \in C_T$. Then, by Theorem 15, $\bar{U} \subset (X - V)$, which means $\bar{U} \cap V = \phi$.

<u>Theorem 17</u>: Let (X,S) and (Y,T) be topological spaces, let F be a lattice isomorphism from S onto T, and let $U \in S$. Then $\overline{F(U)} = \mathcal{D}_F(\bar{U})$.

<u>Proof</u>: Now $\bar{U} \in C_S$ and $(X - U) \in C_S$. Also $\mathcal{D}_F(X - U) = Y - F(X - (X - U)) = Y - F(U)$. Since $U \subset \bar{U}$, and $\bar{U} \cup (X - U) = X$, then

$$Y = D_F(X) = \mathcal{D}_F(\bar{U} \cup (X - U)) = \mathcal{D}_F(\bar{U}) \cup \mathcal{D}_F(X - U) =$$
$$\mathcal{D}_F(\bar{U}) \cup (Y - F(U)).$$

Hence, $F(U) \subset \mathcal{D}_F(\bar{U})$, so by Theorem 16, $\overline{F(U)} \subset \mathcal{D}_F(\bar{U})$.

Now $F(U) \subset \overline{F(U)}$, so $F(U) \cap (Y - \overline{F(U)}) = \phi$. Thus,

$$U \cap [F^{-1}(Y - \overline{F(U)})] = F^{-1}(F(U)) \cap [F^{-1}(Y - \overline{F(U)})] =$$
$$F^{-1}[F(U) \cap (Y - \overline{F(U)})] = F^{-1}(\phi) = \phi.$$

So by Corollary 16.1, $\bar{U} \cap [F^{-1}(Y - \overline{F(U)})] = \phi$, which means $F^{-1}(Y - \overline{F(U)}) \subset (X - \bar{U})$. Thus, $Y - \overline{F(U)} = F[F^{-1}(Y - \overline{F(U)})] \subset F(X - \bar{U})$. Therefore, $\mathcal{D}_F(\bar{U}) = Y - F(X - \bar{U}) \subset Y - (Y - \overline{F(U)}) = \overline{F(U)}$.

Hence, $\overline{F(U)} = \mathcal{D}_F(\bar{U})$.

Theorem 18: Let (X,S) and (Y,T) be topological spaces, let F be a lattice isomorphism from S onto T, and let $U \in S$ and $M \in C_S$ such that $M \subset U$. Then $\mathcal{D}_F(M) \subset F(U)$.

Proof: Since $M \subset U$,

$\phi = \mathcal{D}_F(\phi) = \mathcal{D}_F(M \cap (X - U)) = \mathcal{D}_F(M) \cap \mathcal{D}_F(X - U)$. Thus, $\mathcal{D}_F(M) \subset Y - \mathcal{D}_F(X - U) = Y - [Y - F(X - (X - U))] = Y - [Y - F(U)] = F(U)$.

CHAPTER III

Definition 15: Let (X,T) be a topological space and let $A \subset X$. Then the relative topology for A induced by T, denoted by T_A, is defined to be $\{\mathcal{O} \cap A \mid \mathcal{O} \in T\}$.

Theorem 19: Let (X,T) be a topological space and let $A \subset X$. Then (A,T_A) is a topological space.

Proof: Since $\phi \in T$ and $\phi \cap A = \phi$, then $\phi \in T_A$. Since $X \in T$ and $X \cap A = A$, then $A \in T_A$.

Let $\mathcal{U} \subset T_A$. If $U \in \mathcal{U}$, there exists $\mathcal{O}_U \in T$ such that $\mathcal{O}_U \cap A = U$. Let $Q = \cup \{\mathcal{O}_U \mid U \in \mathcal{U}\}$. Then $Q \in T$, so $Q \cap A \in T_A$. Now let $q \in Q \cap A$. Then $q \in Q$ and $q \in A$. Since $q \in Q$, there is a $U \in \mathcal{U}$ such that $q \in \mathcal{O}_U$. Now $\mathcal{O}_U \cap A = U$. So, $q \in U \in \mathcal{U}$; thus, $q \in \cup\, \mathcal{U}$. Therefore, $Q \cap A \subset \cup\, \mathcal{U}$. Let $p \in \cup\, \mathcal{U}$. There is a $U_1 \in \mathcal{U}$ such that $p \in U_1$. Now $U_1 = \mathcal{O}_{U_1} \cap A$. Hence, $p \in \mathcal{O}_{U_1}$ and $p \in A$. Since $p \in \mathcal{O}_{U_1}$ then $p \in Q$. Thus, $p \in Q \cap A$. Therefore, $\cup\, \mathcal{U} \subset Q \cap A$. Thus, $\cup\, \mathcal{U} = Q \cap A$, and $\cup\, \mathcal{U} \in T_A$.

Let $\mathcal{V} \subset T_A$ such that $\mathcal{V}$ is finite. If $\mathcal{V} = \phi$, then $\cap\, \mathcal{V} = \cap\, \phi = A \in T_A$. Suppose $\mathcal{V} \neq \phi$. If $V \in \mathcal{V}$, then there exists $\mathcal{O}_V \in T$ such that $\mathcal{O}_V \cap A = V$. Let $P = \cap \{\mathcal{O}_V \mid V \in \mathcal{V}\}$. Then $P \in T$, so $P \cap A \in T_A$. Now let $r \in P \cap A$. Then $r \in P$ and $r \in A$. Let $V \in \mathcal{V}$. Then $V = \mathcal{O}_V \cap A$. Since $r \in P$, $r \in \mathcal{O}_V$; and $r \in A$, so $r \in \mathcal{O}_V \cap A = V$. Thus, $r \in \cap\, \mathcal{V}$, and $P \cap A \subset \cap\, \mathcal{V}$. Let $s \in \cap\, \mathcal{V}$.

Since $\mathcal{V} \neq \phi$, there exists a $V_1 \in \mathcal{V}$. Since $V_1 \in T_A$, there is an element $\mathcal{O}_{V_1} \in T$ such that $V_1 = \mathcal{O}_{V_1} \cap A$. Since $s \in \cap \mathcal{V}$, $s \in V_1$ so $s \in A$. Now if $\mathcal{O} \in \{\mathcal{O}_V \mid V \in \mathcal{V}\}$, then there is a $V_2 \in \mathcal{V}$ such that $\mathcal{O}_{V_2} = \mathcal{O}$ and since $s \in \cap \mathcal{V}$, $s \in V_2 = \mathcal{O} \cap A$, so $s \in \mathcal{O}$. Therefore, $s \in \cap \{\mathcal{O}_V \mid V \in \mathcal{V}\} = P$. Thus, $s \in P \cap A$, and $\cap \mathcal{V} \subset P \cap A$. Hence, $\cap \mathcal{V} = P \cap A$. So $\cap \mathcal{V} \in T_A$.

Therefore, T_A is a topology for A, so (A,T_A) is a topological space.

Definition 16: Let (X,T) be a topological space. To say that a set $\mathcal{U}$ is an open cover of X means that $\mathcal{U} \subset T$ and $\cup \mathcal{U} = X$. To say that $\mathcal{V}$ is a subcover of $\mathcal{U}$ means that $\mathcal{V} \subset \mathcal{U}$ and $\mathcal{V}$ is an open cover of X.

Definition 17: Let (X,T) be a topological space. To say that (X,T) is compact means that if $\mathcal{U}$ is an open cover of X, then there exists a subcover $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ is finite.

Definition 18: Let (X,T) be a topological space and let $A \subset X$. To say that A is a compact subset of (X,T) means that (A,T_A) is compact.

Definition 19: Let (X,T) be a topological space. To say that (X,T) is locally compact means that if $p \in X$, then there exists a $U \in T$ with $p \in U$ such that $\bar{U}$ is a compact subset of (X,T).

Theorem 20: Let (X,S) and (Y,T) be topological spaces, let F be a lattice isomorphism from S onto T, and let $D \in C_S$ such that D is a compact subset of (X,S). Then $\mathcal{D}_F(D)$ is a compact subset of (Y,T).

Proof: Let $\mathcal{U}$ be an open cover of $\mathcal{D}_F(D)$. Then if $U \in \mathcal{U}$, there exists an $\mathcal{O}_U \in T$ such that $U = \mathcal{O}_U \cap \mathcal{D}_F(D)$. Let $\mathcal{V} = \{\mathcal{O}_U \mid U \in \mathcal{U}\} \cup (Y - \mathcal{D}_F(D))$. Since $\mathcal{D}_F(D) \in C_T$, $\mathcal{V}$ is an open cover of Y. Let $\mathcal{W} = \{F^{-1}(V) \mid V \in \mathcal{V}\}$. Then $\cup\, \mathcal{W} = \cup \{F^{-1}(V) \mid V \in \mathcal{V}\} = F^{-1}(\cup \{V \mid V \in \mathcal{V}\} = F^{-1}(\cup\, \mathcal{V}) = F^{-1}(Y) = X$. Thus, $\mathcal{W}$ is an open cover of X. Now $F^{-1}(Y - \mathcal{D}_F(D)) = F^{-1}[Y - (Y - F(X - D))] = F^{-1}(F(X - D)) = X - D$. Let $\mathcal{Z} = \{W \cap D \mid W \in \mathcal{W}$ and $W \neq F^{-1}(Y - \mathcal{D}_F(D))\}$. Then if $W \in \mathcal{W} \subset S$, then $W \in S$, so $W \cap D \in S_D$. Also, since $\cup\, \mathcal{W} = X$, then $\cup\, \mathcal{Z} = D$, so $\mathcal{Z}$ is an open cover of D. Since (D,S_D) is compact, there is a finite subcover $\mathcal{R}$ of $\mathcal{Z}$, which covers D. Let $\mathcal{P} = \{W \in \mathcal{W} \mid W \cap D \in \mathcal{R}\} \cup (X - D)$. Then $\mathcal{P}$ is a finite open cover of X. Let $\mathcal{Q} = \{F(P) \mid p \in \mathcal{P}\}$. Then $\cup\, \mathcal{Q} = \cup \{F(P) \mid P \in \mathcal{P}\} = F(\cup \{P \mid P \in \mathcal{P}\}) = F(\cup\, \mathcal{P}) = F(X) = Y$. Thus $\mathcal{Q}$ is a finite open cover of Y. Now $F(X - D) = Y - [Y - F(X - D)] = Y - \mathcal{D}_F(D)$. Let $\mathcal{M} = \{\mathcal{D}_F(D) \cap Q \mid Q \in \mathcal{Q}$ and $Q \neq F(X - D)\}$. Then if $Q \in \mathcal{Q} \subset T$, then $Q \in T$, so $\mathcal{D}_F(D) \cap Q \in T_{\mathcal{D}_F(D)}$. Also, since $\cup\, \mathcal{Q} = Y$, then $\cup\, \mathcal{M} = \mathcal{D}_F(D)$, so $\mathcal{M}$ is a finite open cover of $\mathcal{D}_F(D)$.

Let $M \in \mathcal{M}$. Then there exists a $Q \in \mathcal{Q}$, $Q \neq F(X - D)$, and $M = \mathcal{D}_F(D) \cap Q$. Then there exists a $P \in \mathcal{P}$, $P \neq (X - D)$, such that

$Q = F(P)$. Thus, $M = \mathcal{D}_F(D) \cap F(P)$. Then there exists a $W \in \mathcal{W}$ with $W \cap D \in \mathcal{R}$ and $P = W$. Thus, $M = \mathcal{D}_F(D) \cap F(W)$. Then there exists a $V \in \mathcal{V}$ such that $W = F^{-1}(V)$. Hence, $M = \mathcal{D}_F(D) \cap F(F^{-1}(V)) = \mathcal{D}_F(D) \cap V$. Then there exists a $U \in \mathcal{U}$ such that there exists an $\mathcal{O}_U \in T$ and $U = \mathcal{O}_U \cap \mathcal{D}_F(D)$ and $V = \mathcal{O}_U$. Thus, $M = \mathcal{D}_F(D) \cap \mathcal{O}_U = U \in \mathcal{U}$. Hence $\mathcal{M} \subset \mathcal{U}$.

Therefore, there exists a finite subcover of $\mathcal{U}$, namely $\mathcal{M}$. So $(\mathcal{D}_F(D), T_{\mathcal{D}_F(D)})$ is compact, which means $\mathcal{D}_F(D)$ is a compact subset of (Y,T).

<u>Corollary 20.1</u>: Let (X,S) and (Y,T) be topological spaces, let F be a lattice isomorphism from S onto T, and suppose (X,S) is compact. Then (Y,T) is compact.

<u>Proof</u>: Since $X \in C_S$, and X is a compact subset of (X,S), then $\mathcal{D}_F(X)$ is a compact subset of (Y,T) by Theorem 20. But $\mathcal{D}_F(X) = Y - F(X - X) = Y - F(\phi) = Y - \phi = Y$. So Y is a compact subset of (Y,T), which means $(Y,T) = (Y,T_Y)$ is compact.

<u>Corollary 20.2</u>: Let (X,S) and (Y,T) be topological spaces and let F be a lattice isomorphism from S onto T. Suppose (X,S) is locally compact. Then (Y,T) is locally compact.

<u>Proof</u>: Let $y \in Y$. Let $\mathcal{U} = \{U \mid U \in S$ and $(\bar{U},S_{\bar{U}})$ is compact$\}$. Let $p \in X$. Since (X,S) is locally compact, there exists a $W \in S$ such that $p \in W$ and $(\bar{W},S_{\bar{W}})$ is compact. Thus, $p \in \cup\,\mathcal{U}$ and so $\cup\,\mathcal{U} = X$, which means $\mathcal{U}$ is an

open cover of X. Let $\mathcal{V} = \{F(U) \mid U \in \mathcal{U}\}$. Then $\cup \mathcal{V} = \cup \{F(U) \mid U \in \mathcal{U}\} = F(\cup \{U \mid U \in \mathcal{U}\}) = F(\cup \mathcal{U}) = F(X) = Y$. Then $\mathcal{V}$ is an open cover of Y. Since $y \in Y$, there exists a $V \in \mathcal{V}$ with $y \in V$. But since $V \in \mathcal{V}$, then there is a $U \in \mathcal{U}$ with $V = F(U)$; thus, $\overline{V} = \overline{F(U)} = \mathcal{D}_F(\overline{U})$, by Theorem 17. Hence, $(\overline{V}, T_{\overline{V}}) = (\mathcal{D}_F(\overline{U}), T_{\mathcal{D}_F(\overline{U})})$ is compact, by Theorem 20, which means $\overline{V}$ is a compact subset of (Y,T).

Therefore, if $y \in Y$, then there exists a $V \in T$ with $y \in V$ such that $\overline{V}$ is a compact subset of (Y,T). Hence, (Y,T) is locally compact.

CHAPTER IV

Definition 20: Let (X,T) be a topological space. To say that (X,T) is connected means that there do not exist elements U, V of T such that $U \neq \phi$, $V \neq \phi$, $U \cap V = \phi$, and $X = U \cup V$.

Definition 21: Let (X,T) be a topological space with $A \subset X$. Then to say that A is a connected subset of (X,T) means that (A,T_A) is connected.

Definition 22: Let (X,T) be a topological space, and let $p \in X$. Then to say that (X,T) is locally connected at p means that if $U \in T$ and $p \in U$, then there exists a $V \in T$ such that $p \in V$, $V \subset U$, and V is a connected subset of (X,T).

Definition 23: Let (X,T) be a topological space. To say that (X,T) is locally connected means that if $p \in X$, then (X,T) is locally connected at p.

Theorem 21: Let (X,S) and (Y,T) be topological spaces, let F be a lattice isomorphism from S onto T, and let $U \in S$ such that U is a connected subset of (X,S). Then $F(U)$ is a connected subset of (Y,T).

Proof: Suppose $(F(U), T_{F(U)})$ is not connected. Then there exists $A, B \in T_{F(U)}$ such that $A \neq \phi$, $B \neq \phi$, $A \cap B = \phi$, and $F(U) = A \cup B$. Then there exist $C, D \in T$ such that $A = C \cap F(U)$

and $B = D \cap F(U)$. Then $F^{-1}(C) \in S$. Then
$F^{-1}(A) = F^{-1}(C \cap F(U)) = F^{-1}(C) \cap F^{-1}(F(U)) = F^{-1}(C) \cap U \in S_U$.
Similarly, $F^{-1}(B) \in S_U$. Now since $A \cap B = \phi$, then
$F^{-1}(A) \cap F^{-1}(B) = F^{-1}(A \cap B) = F^{-1}(\phi) = \phi$. Also, since $F(U) = A \cup B$, then $F^{-1}(A) \cup F^{-1}(B) = F^{-1}(A \cup B) = F^{-1}(F(U)) = U$. Thus, (U,S_U) is not connected, which is impossible.

Hence, $(F(U), T_{F(U)})$ is connected.

Corollary 21.1: Let (X,S) and (Y,T) be topological spaces, and let F be a lattice isomorphism from S onto T. Suppose (X,S) is connected. Then (Y,T) is connected.

Proof: Since X is an open, connected subset of (X,S), then by Theorem 21, $F(X)$ is a connected subset of (Y,T). But $F(X) = Y$. Hence, $(Y,T) = (Y,T_Y)$ is connected.

Corollary 21.2: Let (X,S) and (Y,T) be topological spaces, and let F be a lattice isomrophism from S onto T. Suppose (X,S) is locally connected. Then (Y,T) is locally connected.

Proof: Let $y \in Y$. Let $V \in T$ such that $y \in V$. Let $W = F^{-1}(V)$. Then by Theorem 13, there is a $q \in W$ such that if $U \in S$ and $q \in U$, then $y \in F(U)$. Since (X,S) is locally connected, there exists an $\mathcal{O} \in S$ such that $\mathcal{O} \subset W$, $q \in \mathcal{O}$, and $\mathcal{O}$ is a connected subset of (X,S). By Theorem 21, $F(\mathcal{O})$ is a connected subset of (Y,T). By Theorem 13, $y \in F(\mathcal{O}) \in T$. Since $\mathcal{O} \subset W$, $F(\mathcal{O}) \subset F(W) = F(F^{-1}(V)) = V$. Thus, (Y,T) is locally connected at y, so (Y,T) is locally connected.

CHAPTER V

<u>Definition 24</u>: Let (X,T) be a topological space, and let $A \subset X$. To say that A is <u>dense</u> means that if $U \in T$ and $U \neq \phi$, then there exists a $p \in A$ such that $p \in U$.

<u>Definition 25</u>: Let (X,T) be a topological space. To say that (X,T) is <u>separable</u> means that there exists a $D \subset X$ such that D is dense and countable.

<u>Definition 26</u>: Let (X,T) be a topological space. To say (X,T) is <u>first countable</u> means that if $p \in X$, then there is a countable subset $\mathcal{U}$ of T such that if $\mathcal{O} \in T$ and $p \in \mathcal{O}$, then there exists a $U \in \mathcal{U}$ with $p \in U \subset \mathcal{O}$. If p is an element of every element of $\mathcal{U}$, then $\mathcal{U}$ is said to be a <u>countable base for T at p</u>.

<u>Definition 27</u>: Let (X,T) be a topological space. To say that (X,T) is <u>second countable</u> means that there exists a countable subset $\mathcal{U}$ of T such that if $p \in X$ and $\mathcal{O} \in T$ and $p \in \mathcal{O}$, then there is a $U \in \mathcal{U}$ such that $p \in U \subset \mathcal{O}$. The collection $\mathcal{U}$ is said to be a <u>countable base for</u> T.

<u>Definition 28</u>: To say that the ordered pair (X,M) is a <u>well-ordered set</u> means that X is a set, and M is a subset of $X \times X$ such that:

1. If $x, y \in X$, then either $(x,y) \in M$ or $(y,x) \in M$,
2. If $x, y \in X$, $(x,y) \in M$, and $(y,x) \in M$, then $x = y$,

3. If $x, y, z \in X$, $(x,y) \in M$, and $(y,z) \in M$, then $(x,z) \in M$, and

4. If $A \subset X$ and $A \neq \phi$, then there is an element $x \in A$, called the <u>least element of</u> A, such that if $a \in A$, then $(x,a) \in M$.

The set M is called a <u>well-ordering</u> for X.

<u>Theorem 22</u>: There is an uncountable set A and a well-ordering $\leq$ for A such that if $a \in A$, then $\{x \in A \mid x < a\}$ is countable.

<u>Proof</u>: Let L be an uncountable set and let $<^*$ be a well-ordering for L. Let $K = L \cup \{L\}$. Let $<<$ be a set such that $(a,b) \in <<$ if and only if $a, b \in K$ and either $a <^* b$, or $b = L$. Then $<<$ is a well-ordering for K. Let
$C = \{x \mid x \in K$ and $\{y \mid y \in K, y \neq x$, and $y << x\}$ is uncountable$\}$. Then $L \in C$, thus $C \neq \phi$. Since $C \subset K$ and $C \neq \phi$, then there exists a $p \in C$ such that p is the least element of C.

Let $A = \{x \mid x \in K, x \neq p$ and $x << p\}$. Since $p \in C$, then A is uncountable. Let $\leq$ be a set such that $(a,b) \in \leq$ if and only if $a, b \in A$ and $a << b$. Then $\leq$ is a well-ordering for A. Since p is the least element of C, then A and $\leq$ have the required properties.

For the remainder of Chapter V, the set A and the well-ordering $\leq$ for A described in Theorem 22, will be referred to as ω and $\leq$, respectively. Also, the set $\omega \cup \{\omega\}$ will be referred to, with the well-ordering $\leq$ defined by if $a,b \in \omega \cup \{\omega\}$, then $a \leq b$ if and only if either $a,b \in \omega$ and $a \leq b$ or $b = \omega$.

Example 1: Let $X = \omega$ and let $Y = \omega \cup \{\omega\}$. Let S and T be the topologies for X and Y, respectively, defined by

$S = \{\phi\} \cup \{X\} \cup \{\{y \mid y \in X \text{ and } x < y\} \mid x \in X\}$, and

$T = \{\phi\} \cup \{Y\} \cup \{\{y \mid y \in Y \text{ and } x < y\} \mid x \in X\}$. Define $F: S \to T$ by $F(\phi) = \phi$, $F(X) = Y$, and if $x \in X$, then

$F(\{y \mid y \in X \text{ and } x < y\}) = \{y \mid y \in Y \text{ and } x < y\}$. Then F is a lattice isomorphism from S onto T, (X,S) is not separable and is first countable, but (Y,T) is separable and is not first countable.

Proof: Clearly, F is one-to-one and onto. Let $\mathcal{U} \subset S$ with $\mathcal{U} \neq \phi$. For each $U \in \mathcal{U}$, let x_U be the element of X such that $U = \{y \mid y \in X \text{ and } x_U < y\}$. Let h be the least element of $\{x_U \mid U \in \mathcal{U}\}$. Then

$$F(\cup \{U \mid U \in \mathcal{U}\}) = F(\cup \{\{y \mid y \in X \text{ and } x_U < y\} \mid U \in \mathcal{U}\}) =$$
$$F(\{y \mid y \in X \text{ and } h < y\}) = \{y \mid y \in Y \text{ and } h < y\} =$$
$$\cup \{\{y \mid y \in Y \text{ and } x_U < y\} \mid U \in \mathcal{U}\} =$$
$$\cup \{F(\{y \mid y \in X \text{ and } x_U < y\}) \mid U \in \mathcal{U}\} = \cup \{F(U) \mid U \in \mathcal{U}\}.$$

Hence, F is a similarity mapping. Hence, by Theorem 5, F is a lattice isomorphism.

Suppose (X,S) is separable. Then there is a countable dense subset D of X. Let $\mathcal{D} = \cup \{\{x \mid x \in X \text{ and } x < d\} \mid d \in D\}$. Since for each $d \in D$, $\{x \mid x \in X \text{ and } x < d\}$ is countable, $\mathcal{D}$ is countable. Since X is uncountable, there is a $p \in X$ such that $p \notin \mathcal{D}$ and there is a $q \in X$ such that $p < q$. Hence, $q \notin \mathcal{D}$. Then $\{x \mid x \in X \text{ and } p < x\}$ is an element of S which contains q and no element of D. Hence, D is not dense, which is impossible. Thus, (X,S) is not separable.

Since $\{\omega\}$ is a countable, dense subset of (Y,T), then (Y,T) is separable.

Suppose $\mathcal{V}$ is a countable base for T at ω. For each $V \in \mathcal{V}$, let y_V be the element of Y such that $V = \{y \mid y \in Y \text{ and } y_V < y\}$. Then let $E = \cup\, \{\{y \mid y \in Y \text{ and } y < y_V\} \mid V \in \mathcal{V}\}$. Since for each $V \in \mathcal{V}$, $\{y \mid y \in Y \text{ and } y < y_V\}$ is countable, then E is countable. Since Y is uncountable, there is an $r \in Y$ such that $r \notin E$ and $r \neq \omega$, and there is an $s \in Y$ such that $r < s < \omega$. Then $\{y \mid y \in Y \text{ and } s < y\}$ is an element of T which contains ω and yet contains no element of $\mathcal{V}$. Hence, there is no countable base for T at ω. Thus, (Y,T) is not first countable.

Let $z \in X$. Then $\{\{y \mid y \in X \text{ and } x < y\} \mid x \in X \text{ and } x < z\}$ is a countable base for S at z. Hence, (X,S) is first countable.

<u>Theorem 23</u>: Let (X,S) and (Y,T) be topological spaces and let F be a lattice isomorphism from S onto T. Suppose (X,S) is second countable. Then (Y,T) is second countable.

<u>Proof</u>: Let $\mathcal{B}$ be a countable base for S. Let $\mathcal{D} = \{F(B) \mid B \in \mathcal{B}\}$. Clearly, $\mathcal{D}$ is countable. Let $p \in Y$ and let $\mathcal{O} \in T$ with $p \in \mathcal{O}$. Then $F^{-1}(\mathcal{O}) \in S$. Then there is a subset $\mathcal{E}$ of $\mathcal{B}$ such that $F^{-1}(\mathcal{O}) = \cup\, \{E \mid E \in \mathcal{E}\}$. Then $\mathcal{O} = F(F^{-1}(\mathcal{O})) = F(\cup\, \{E \mid E \in \mathcal{E}\}) = \cup\, \{F(E) \mid E \in \mathcal{E}\}$. Thus, there

exists an $E \in \mathcal{E}$ such that $p \in F(E) \subset \mathcal{O}$. Since $E \in \mathcal{E} \subset \mathcal{B}$, then $F(E) \in \mathcal{D}$. Hence $\mathcal{D}$ is countable base for T, and (Y,T) is second countable.

CHAPTER VI

Definition 29: Let (X,T) be a topological space. To say that (X,T) is a T_0-space means that if $p, q \in X$ and $p \neq q$, then there exists a $U \in T$ such that either $p \in U$ or $q \in U$, but both p and q are not elements of U.

Definition 30: Let (X,T) be a topological space. To say that (X,T) is a T_1-space means that if $p, q \in X$ and $p \neq q$, then there exists $U, V \in T$ such that $p \in U$ and $q \notin U$ and $q \in V$ and $p \notin V$.

Definition 31: Let X be a non-empty set. A function d from $X \times X$ into the reals is said to be a metric on X provided:

1. If $x \in X$, then $d(x,x) = 0$ and if $x, y \in X$ and $d(x,y) = 0$, then $x = y$,
2. If $x, y \in X$, then $d(x,y) = d(y,x)$, and
3. If $x, y, z \in X$, then $d(x,y) + d(y,z) \geq d(x,z)$.

Definition 32: Let (X,T) be a topological space. To say that (X,T) is a metric space means that there is a metric d on X such that if $U \in T$ and $p \in U$, then there is an $r > 0$ and $\{x \mid d(p,x) < r\} \subset U$.

Example 2: Let $X = \{1\}$ and $S = \{X,\phi\}$. Let $Y = \{1,2\}$ and $T = \{Y,\phi\}$. Define $F:S \to T$ by $F(\phi) = \phi$ and $F(X) = Y$. Then F is a lattice isomorphism from S onto T, (X,S) is a metric space, but (Y,T) is not even a T_0-space.

Proof: Clearly, this is true.

Definition 33: Let (X,S) and (Y,T) be topological spaces, and let f be a function from X into Y. To say that f is open means that if $U \in S$, then $f(U) = \{f(x) \mid x \in U\}$ is an element of T.

Definition 34: Let (X,S) and (Y,T) be topological spaces, and let f be a function from X into Y. To say that f is continuous means that if $V \in T$, then $f^{-1}(V) = \{x \mid f(x) \in V\}$ is an element of S.

Definition 35: Let (X,S) and (Y,T) be topological spaces. To say that a function f is a homeomorphism from (X,S) onto (Y,T) means that f is one-to-one, has domain X and range Y, is open, and is continuous. To say that (X,S) is homeomorphic to (Y,T) means that there exists a homeomorphism from (X,S) onto (Y,T).

Theorem 24: Let (X,S) and (Y,T) be topological spaces, and let f be a homeomorphism from (X,S) onto (Y,T). Then S is lattice isomorphic to T.

Proof: Define a function $F:S \to T$ by if $U \in S$, then $F(U) = f(U)$. If $U, V \in S$ and $F(U) = F(V)$, then $U = f^{-1}(f(U)) = f^{-1}(F(U)) = f^{-1}(F(V)) = f^{-1}(f(V)) = V$. Hence, F is one-to-one.

If $W \in T$, then since f is continuous, $f^{-1}(W) \in S$, and $F(f^{-1}(W)) = f(f^{-1}(W)) = W$. Thus, F has domain S and range T.

If $U, V \in S$ and $U \subset V$, then $F(U) = f(U) \subset f(V) = F(V)$.

If $C, D \in T$ and $C \subset D$, then
$F^{-1}(C) = f^{-1}(C) \subset f^{-1}(D) = F^{-1}(D)$.

Thus, F is a lattice isomorphism from S onto T; hence S is lattice isomorphic to T.

Example 3: Let $X = \{1\}$ and $S = \{X, \phi\}$. Let $Y = \{1,2\}$ and $T = \{Y, \phi\}$. Define $F:S \to T$ by $F(\phi) = \phi$ and $F(X) = Y$. Then F is a lattice isomorphism from S onto T, but (X,S) is not homeomorphic to (Y,T).

Proof: Clearly, this is true.

Theorem 25: Let (X,T) be a T_1-space and let $p \in X$. Then $\{p\} \in C_T$.

Proof: Let $q \in X - \{p\}$. Then since (X,T) is a T_1-space there exists a $U \in T$ such that $q \in U$ and $p \notin U$. Thus, q is not a limit point of $\{p\}$. Hence, by Theorem 14, $\{p\} \in C_T$.

Theorem 26: Let (X,S) and (Y,T) be T_1-spaces, and let F be a lattice isomorphism from S onto T. Let $x \in X$. Then $\mathcal{D}_F(\{x\})$ has exactly one element.

Proof: Since $\{x\} \neq \phi$, then $\mathcal{D}_F(\{x\}) \neq \phi$. Let $p \in \mathcal{D}_F(\{x\})$. Since (Y,T) is a T_1-space, $\{p\} \in C_T$. Since $\{p\} \subset \mathcal{D}_F(\{x\})$, then $\mathcal{D}_F^{-1}(\{p\}) \subset \mathcal{D}_F^{-1}(\mathcal{D}_F(\{x\})) = \{x\}$. Since $\mathcal{D}_F^{-1}(\{p\}) \neq \phi$, $\mathcal{D}_F^{-1}(\{p\}) = \{x\}$. Thus, $\{p\} = \mathcal{D}_F(\mathcal{D}_F^{-1}(\{p\})) = \mathcal{D}_F(\{x\})$. Thus, $\mathcal{D}_F(\{x\})$ has exactly one element.

Theorem 27: Let (X,S) and (Y,T) be T_1-spaces and let F be a lattice isomorphism from S onto T. Then (X,S) is homeomorphic to (Y,T).

Proof: Define a function $f:X \to Y$ as follows: If $x \in X$, by Theorem 26, $\mathcal{D}_F(\{x\})$ has exactly one element y; then define $f(x) = y$.

If $x, z \in X$ with $f(x) = f(z)$, then

$\{x\} = \mathcal{D}_F^{-1}(\mathcal{D}_F(\{x\})) = \mathcal{D}_F^{-1}(\{f(x)\}) = \mathcal{D}_F^{-1}(\{f(z)\}) = \mathcal{D}_F^{-1}(\mathcal{D}_F(\{z\})) = \{z\}$, and thus $x = z$. Hence, f is one-to-one.

Let $y \in Y$. Then $\{y\} \in C_T$, so $\mathcal{D}_F^{-1}(\{y\}) \in C_S$. Let p be the element of $\mathcal{D}_F^{-1}(\{y\})$. Then

$\{f(p)\} = \mathcal{D}_F(\{p\}) = \mathcal{D}_F(\mathcal{D}_F^{-1}(\{y\})) = \{y\}$. Hence, $f(p) = y$ and f has domain X and range Y.

Let $U \in S$. If $x \in U$, then $\{x\} \subset U$ so $f(x) \in \mathcal{D}_F(\{x\}) \subset F(U)$ by Theorem 18. Thus, $f(U) = \{f(x) \mid x \in U\} \subset F(U)$. Now let $z \in F(U)$. Then $\{z\} \subset F(U)$, so $\mathcal{D}_F^{-1}(\{z\}) \subset F^{-1}(F(U)) = U$. Now there exists an $a \in X$ such that $\{a\} = \mathcal{D}_F^{-1}(\{z\})$ thus $a \in U$ and $f(a) = z \in F(U)$. Thus, $z \in \{f(x) \mid x \in U\} = f(U)$. Thus, $F(U) \subset f(U)$. Hence $f(U) = F(U) \in T$. So f is open.

Let $V \in T$. If $x \in X$ such that $f(x) \in V$, then $\mathcal{D}_F(\{x\}) = \{f(x)\} \subset V$, and $\{x\} = \mathcal{D}_F^{-1}(\mathcal{D}_F(\{x\})) \subset F^{-1}(V)$, so $x \in F^{-1}(V)$. Thus, $f^{-1}(V) = \{x \mid f(x) \in V\} \subset F^{-1}(V)$. Now if $x \in F^{-1}(V)$, then $\{x\} \subset F^{-1}(V)$, so $f(x) \in \mathcal{D}_F(\{x\}) \subset F(F^{-1}(V)) = V$. Thus, $F^{-1}(V) \subset \{x \mid f(x) \in V\} = f^{-1}(V)$. Hence, $f^{-1}(V) = F^{-1}(V) \in S$. So f is continuous.

Thus, there exists a function $f:X \rightarrow Y$ such that f is one-to-one, onto, open and continuous. Hence, (X,S) is homeomorphic to (Y,T).

SUMMARY

In this thesis the ideas of lattice, lattice isomorphism, and similarity have been investigated. Specifically, these concepts have been related to topologies. It has been shown that compact, locally compact, connected, locally connected, and second countable are preserved by lattice isomorphisms; while separable, first countable, and separation axioms are not.

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